How to Play in Tune on Stringed Instruments

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What a title! As if all which might be learned on this topic could be incorporated into one brief article when performers spend a lifetime perfecting this skill. Nevertheless, there are certain scientific acoustical principles which can be applied to launch the student on his or her way toward that goal. This article is oriented toward violin players. Simply apply by analogy the relevant information to the viola and cello fingerboards.

In the discussions of various notes below the open strings are indicated as "E, A, D," and "G" but the actual notes discussed are in bold: **a**, **b**, **C**, etc. The absolute pitch levels of the notes are indicated in the usual way: **C** 2nd line blow the bass clef; **c** 2nd space of the bass clef; **c**', middle c one ledger line below the treble clef; **c**" 3rd space of the treble clef; **c**'" 2nd ledger line above the treble clef, etc.

First of all, imagine that instead of a violin you are playing a guitar. Those helpful frets on a guitar relieve the performer of using his or her ear to determine those micro accurate left hand finger placements. But those micro accurate ear-finger judgments make the difference between good and bad intonation on a fretless instrument. And as we all know it's quite a job to accomplish this without those frets.

(As a matter of fact, back in 1756, Leopold Mozart, in his book on violin playing, advised the teacher against carving notches in the fingerboard to help the students find those finger placements!) In order to conceive of the violin as a guitar your imagination must supply those missing frets; and in your mind each of those frets is assigned a number as your left hand fingers go up the fingerboard. Instead of fret numbers I assign to myself and my students "half-step numbers."

For example: the open strings are at the "zero half step." Besides being known as G,D,A, and E they are at times also called **f**x, **a**bb; **c**x', **e**bb'; **g**x', **b**bb'; **d**x", and **f**b". The first fingered half step on the G string is known as **g#** or **a**b; and its companions on the first half step going rightward across the fingerboard are called respectively **d#**' or **e**b'; **a**#' or **b**b'; and **e**#" and **f**". The second half step, again going from G string to E string are: **a**, **g**x or **b**bb; **e'**, **d**x', or **f**b'; **b'**, **a**x', or **c**b"; and **f"**#, **e**x", or **g**b". And so it goes up the fingerboard until we reach the 12th half step at which point the string is divided in half by the first natural harmonics, **g**',**d**",**a**", and **e**'". I don't usually use my numbering system above the 12th half step simply for practicality but of course the numbers could be continued farther if desired. On the piano, as we know, all those enharmonic equivalents mentioned above which match one another are played with the same key. But for the violinist it's not so easy.

While many of us practice just to get those particular notes into their proper places at the right time it just so happens that there are at least twice as many audible and discernible pitches and pitch variations on those twelve half steps than we would like to think; and our job to play in tune requires twice as much work as seems necessary at first glance. The purpose of this article is to list the techniques for finding all those various pitch variations at each particular half step. Most half steps have at least two versions: a high version and a low version. Some of the half steps have three: a high version, a middle version, and a low version; and one half step, the 6th, has four. These can all be found by tuning the fingers both to various open strings and/or to previously established tuned finger placements: "preliminary tunings."

Good intonation is based on, but not exclusively restricted to, the pure, tunable intervals. What are the pure intervals? "Interval" means "the distance between," in this case the acoustical space between two different notes. That acoustical space is measured by the vibration frequency of each particular pitch and the differences in numbers of vibrations between the desired pitches. (Back in the days of the ancient musico-philosophizing Greeks some 3,000 years ago when frequencies were not countable, some of those early Greek acousticians such as Pythagoras came to these same conclusions about acoustical spaces by measuring different lengths of vibrating strings of the same material, thickness, and tension and noting what intervals resulted.)

As we all know, "A" is "440." That is, if a string vibrates 440 times per second we will all hear a pitch we have decided by convention to call "A,": the open A string on a violin or viola. And as some of us also know, if a string had vibrated at 440 250 years ago most musicians of those days would more likely have called it "Bb." These pitch naming conventions as well as the absolute levels of the pitches themselves have changed over the centuries, the actual pitches usually in an upward direction, the names used for them in an alphabetically lower direction.

When two or more pitches are bowed simultaneously on a stringed instrument we call this process "playing double stops." But obviously if one or more of the played notes are open strings then these are not "stopped" at all; so the concept is a misnomer but we're stuck with it. If two strings of different pitches are bowed simultaneously then the ear will hear not only the pitches being bowed but one or more "difference tones" at the same time. What are the difference tones and how are they produced? And what do they have to do with perfecting one's intonation?

Sound waves alternate moments of increased air pressure ("condensation") with moments of decreased air pressure ("rarefaction"). If a continuous series of sound waves are regular and not random then our ears define these series as specific pitches which differ by the frequency of these alternations, just as we name variations in frequencies of light waves by different color names. Difference tones are caused by two simultaneously sounding original pitches with regular but differing alternations of condensation and rarefaction. If two moments of condensation of the two pitches occur at the same time then the increased air pressure forms a kind of thump in our ears.

If on the other hand two moments of rarefaction coincide then a fading away of the air pressure results. This alternation of thumping and fading at mathematically regular moments creates the difference tones. If the two original pitches are close in frequency then the difference tones will take the form of slow beats, both perceptible and countable. But the farther apart in frequency the original pitches become, the faster the beats also become and soon they are no longer countable to our ears but transform themselves by their increasing frequency into a buzz.

Now since this buzz too has a regular frequency it also becomes a recognizable pitch to our ears. And, as we know, the faster the frequency of a regularly vibrating object becomes, the higher the pitch into which it translates in our ears also becomes. Thus the farther apart the two original pitches become, (i.e., the greater the number of the difference of their vibrating frequencies) the higher the pitch of the difference tone we perceive.

For example: the quintessential interval without beats is the unison. Since both of the notes which make up this "interval" (There is no acoustical space between the two notes; hence the quotation marks.) have the identical frequency, their individual moments of condensation and rarefaction are always simultaneous; thus no beats. But have one note vibrating at 440 and the other, a lower pitched note, at 439, then one beat per second will be audible: 440-439=1 when the two are played together. Lower the second note to 438 and two beats per second will become audible and still countable; 437=3; to 436=4, etc.

Lower, however, the second note sufficiently, say to 400, and the beats will turn into a buzz whose frequency is 40 per second. No one can count 40 beats per second but a certain pitch whose frequency is 40 per second has become perceptible. The slower the beat count the lower its pitch seems to our ears; if the beats are too few they are not perceived as a pitch at all except perhaps by those accustomed to very low notes on the organ or the doublebass. We will address the mathematical relationships between the beat numbers and their producing notes below.

It was no coincidence that the first musician to perceive and identify the "difference tones" was a violinist, namely Giuseppe Tartini, in the 18th century; and for some time after his discoveries these phenomena were known as the "Tartini tones." Why a violinist rather than a harpsichordist, organist, singer, or flutist? The reasons are several: (1) In order for the difference tones to be perceived two tones must be performed simultaneously as on two different strings of the same string instrument; (2) the mathematical relation of the two frequency numbers must be easily measurable, the ratio of their frequencies fractionally reducible to the lowest possible terms (See below.); (3) the persistence in the air of the two tones must remain at sufficient continuity and formidable strength (i.e. be loud enough); and (4) their pitch levels must be such that both the original tones and the resultant difference tones fall within the normal individual's hearing range. The violin fulfills these conditions more perfectly than any other instrument. Among the keyboard instruments only the organ might have come close. But its more and more tempered way of being tuned as the centuries progressed prevented it from producing the mathematically measurable intervals necessary.

On keyboard instruments, (except for the organ, either acoustic or electronic, which can sustain the tones), while two notes may be struck simultaneously the usual quick decay time of the performed tones (as on a harpsichord or piano) makes the perception of the difference tones more difficult. The tempering problem as described above stands equally in the way. Singers and non-string instrumentalists would need a partner to produce the simultaneity of the performed tones (except for the clarinettist's "multiphonics"). A string instrument would need to be bowed continuously rather than merely struck or plucked (like a zither or a guitar) so that the difference tones might appear and become perceptible. Bowed instruments can also be played loudly, also a necessary condition for perceiving and identifying the difference tones. Finally, the range of the violin (but not that of the viola, cello, or double bass) permits the ear to more easily perceive the difference tones, which usually form deep in the bass.

Now it's time for a bit of heavy hitting: musical fractions, otherwise known as ratios. In acoustics when two pitches are compared their individual frequencies are juxtaposed in the form of a fraction with the higher note of greater frequency usually being put in place as a numerator and the lower note of lesser frequency set as a denominator. Then mathematical observations are made defining the relationship of the two notes from the "double stop" point of view.

For example, in the unison mentioned above if both notes happened to be our familiar friend **a**' at 440, say in tuning up between two violinists, or one performer checking a fingered **a**' with the open A string, then 440 would serve as both numerator and denominator in our acoustical fraction=440/440. Our goal in acoustical manipulations is to reduce such fractions to their lowest possible terms. This makes their true intervallic relationship as clear as possible. To get the lowest possible terms you find the highest possible identical number which will divide both numerator and denominator successfully and successively.

In this case 440 itself is the highest such number. 440 the numerator divided by 440=1; and 440 the denominator also divided by 440=1 as well so that the resulting fraction in its lowest possible terms is revealed as 1/1. So whether the frequency of any two equal pitches be 10/10 or 1,000,000/1,000,000 the resulting fractional reduction will always give us the identical result=1/1. Thus in acoustical terms we say, "The ratio (fraction) of the unison is 1/1."

Another example: to get an octave above our A at 440 we must play the octave **a**" on the E string whose frequency is 880. The resulting fraction is thus 880/440. The highest number which will divide (or as we say "go into") both numerator and denominator evenly is not 880 but 440. This operation gives us the lowest terms of this fraction as 2/1. Thus “the ratio of the octave is 2/1.” This ratio expresses the frequency relationship of any and all octaves of whatever frequencies. The lower octave to our A 440 would be the **a** first or half position on the G string. Its frequency is 220. So this octave’s fraction would be 440/220. Dividing both numerator and denominator by 220 would give us the identical 2/1, etc.

So what would be the resulting difference tone if we played both notes of our octave simultaneously? Exactly half of the time the frequencies of the two notes would be interfering with one another in the way described above, sometimes reinforcing each other in strength, sometimes in weakness, creating the thump or buzz mentioned. When the condensation of one frequency coincides with the rarefaction of the other they cancel each other out and the moment becomes “thumpless.” Thus the thump/buzz is caused by the alternation of the reinforcing moments with the canceling-out moments. These alternations too can be mathematically predictable and our difference tones perceived by the creation of yet another fraction.

We take the ratio of our interval, here 2/1, and subtract the denominator from the numerator. In this case the result would be [1]. We take this difference number and set it below the original denominator as a new denominator in a three-part faction, here 2/1/1. Then we examine the new ratios created between these three numbers by setting the remainder number against either of the two original numbers to create a new two-part fraction. This will be the ratio of the perceptible difference tone. And by examining this ratio we can determine the interval of the difference tone to either of the two original tones.

Thus our resulting [1] when set as a new denominator under the original numerator gives us 2/1. Setting it as a denominator to the original denominator results in 1/1. So we see that the difference tone of an octave is an octave below the upper note of our original two tones and a unison to the lower. In the case of the octave the difference tone is identical to one of the two original tones and is therefore imperceptible as a separate tone. But it does strengthen the lower note. I still remember one of my earlier teachers, Mischa Mischakoff, always telling me, “The lower note of an octave [double-stopped] always sounds stronger than the upper one.” After some years of acoustical study I discovered the reason for this phenomenon which I’ve described here.

One further example: when we play the A string and the E string together we get the frequencies 440 and 660, or in our usual form 660/440. To reduce this fraction to its lowest terms we find that the highest number which will divide both numerator and denominator evenly is neither 660 or 440 but 220. The resulting division/reduction gives us 3/2. Thus, “the ratio of a perfect 5^{th} is 3/2.” 3-2=1. So our new 3-part fraction is 3/2/1.

While the relationship of our difference number to the original numerator is not immediately apparent (Techniques for this comparison will be shown below.) we do see immediately that the new 2-part fraction between the original denominator and our difference number is 2/1. This tells us that the difference tone of a perfect 5th is an octave below the lower note. So that if your two original notes are indeed the A and E strings your difference tone will be the pitch of the lower octave, **a** first/half position on the G string. (The difference tones of fingered 5ths are easier to hear, however.) When going for difference tones remember that the higher your two original pitches the easier your difference tones will be to hear, and vice versa. This has to do with the hearing capacity and range of the human ear.

Now that the principle of frequency ratios has been illustrated it will no longer be necessary to cite actual frequencies of any particular pitches when describing intervals between them. These intervals can now be referred to by ratio only.

Some of the more common interval ratios are the following:

Unison	1/1
Octave	2/1
Perfect 5th	3/2
Perfect 4th	4/3
Major 3rd	5/4
Minor 3rd	6/5
Large Major 2nd	9/8
Small Major 2nd	10/9
Minor 2nd	16/15
Major 6th	5/3
Minor 6th	8/5
Minor 7th	9/5
Large Major 7th	15/8
Small Major 7th	16/9

The missing ratios "7/6" and "8/7" are unusable in Western music. They lie somewhere between a large major 2nd and a Minor 3rd.

Sometimes the resulting difference number after the appropriate subtraction is too low to fit clearly into any of the above ratios. Such as:

4/3/1, the difference tone of a perfect 4th. We simply double the difference number so as to create the octave above it. This doubling of 1 to "2" gives us a number we can compare to the parts of our original ratio: 4/"2" or 3/"2". This tells us that the difference tone of a perfect 4th is an octave plus an octave (2 octaves) below the upper note or a perfect 5th plus an octave below the lower note. The two octaves mentioned in the original numerator comparison were first of all the reduction of 4/2 to 2/1 plus the subtracting back of the octave we supplied for clarification. In the original denominator comparison we subtracted back the supplied octave.

So if we're playing the 7th half step on the A string (**e**") against the 5th half step on the E string (**a**") (1st, 2nd, or 3rd position) our difference tone would be a perfect 5th plus an octave below the played **e**" or two octaves below the played **a**". In both cases it's the same note, 1st or half position **a** on the G string. That's the note we would hear sounding in our ears along with the played notes, **e**" and **a**". However, many of the difference tones on a violin fall below the normal violin range. The lower the original two played notes the harder it is for most of us violinists to hear the difference tones which appear deep in the bass.

And now having supplied the reader with sufficient acoustical background for extracting the difference tones from double stops of the pure intervals it's time to turn to the individual half steps and explain how to tune each one up. But before we can do that we have to describe still another historical circumstance: the battle which has been going on for centuries (perhaps two or three millennia) between two camps each of which prefers a different set of acoustical intonation preferences, the one irreconcilable to the other, namely, the battle between the Pythagoreans and the Tartinians and how the well-tuned violin player has to pick his way carefully between the two camps. Theodore Podnos summed up this historical-acoustical battle quite admirably in his highly illuminating book, *Intonation for Strings, Winds, and Singers*, (Metuchen NJ, 1981).

The battle began 3,000 years ago when Pythagoras, mentioned above, decided for whatever philosophic reasons, that all musical pitches had to be derived from the perfect 5th. So if you started from C then your next G upward had to tune to the original C at the ratio of 3/2. Then the next note up, D, likewise 3/2 from G, etc. until all 12 different notes were accounted for around the "circle of 5^{th}s." (That's how we tune our 4 strings, be they G,D,A,E or C,G,D,A. Perhaps the musical Pythagoras was the same person as the geometrical Pythagoras. This would have explained his liking for basic fractions.)

But let Wikipedia continue:

"Pythagorean comma

From Wikipedia, the free encyclopedia

In music, when ascending from an initial (low) pitch by a cycle of justly tuned perfect fifths (ratio 3:2), leapfrogging twelve times, one eventually reaches a pitch approximately seven whole octaves above the starting pitch. If this pitch is then lowered precisely seven octaves, it will be discovered that the resulting pitch is 23.46 cents (a quarter of an equal tempered minor second) higher than the initial pitch. This microtonal interval

$$\left(\frac{3}{2}\right)^{12} \Big/ 2^7 = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} = 1.0136432647705078125$$

is called a **Pythagorean comma**, named after the ancient mathematician and philosopher Pythagoras. It is sometimes called a **ditonic comma**.

Put more succinctly, twelve perfect fifths are not exactly equal to seven perfect octaves, and the Pythagorean comma is the amount of the discrepancy.

This interval has serious implications for the various tuning schemes of the chromatic scale, because in Western music, 12 perfect fifths and seven octaves are treated as the same interval. Equal temperament, today the most common tuning system used in the West, accomplished this by flattening each fifth by a twelfth of a Pythagorean comma (2 cents), thus giving perfect octaves.

Chinese mathematicians had been aware of the Pythagorean comma as early as 122 BC (its calculation is detailed in the *Huainanzi*), and circa 50 BC, Ching Fang discovered that if the cycle of perfect fifths were continued beyond 12 all the way to 53, the difference between this 53rd pitch and the starting pitch would be much smaller than the Pythagorean comma, which was later named Mercator's comma. (*see: history of 53 equal temperament*).

Other intervals of similar size are the syntonic comma, and Holdrian comma." [end of Wikipedia article]

Never mind all those decimal fractions and their "cents." Numerical fractions using whole numbers for numerators and denominators are usually enough for string players who have to deal with musico-acoustical problems. We'll deal with an even more important string player's "comma" later on, the so-called "Comma of Didymus" (the "syntonic comma" as named above) which will be explained further along. "Commas", "limmas," and "schismas" are different Greek terms used to name those different microintervals resulting from these acoustical calculations, comparisons, and non-congruences. And speaking of "53" as Wikipedia was doing, there once was a piano constructed in which all 53 of those microgradations were represented by tuned strings within one octave. As we work our way up the fingerboard along those 12 half steps we'll have slightly more than half that number to figure out and tune up. However, the above-described non-identity between 12 perfect 5^{th}s and seven octaves is more a problem for piano tuners rather than for string players. Our basic problem shows up on the circle of fifths well before then and concerns the question of the tuning of the major 3^{rd}.

Take the open G string up to **b**. Theoretically the ratio of this major 3rd is 5/4 (5=**b**, 4=**g**) *a la* Tartini with his difference-tone producing "just" intervals. (A "just" interval is one produced by those basic fractions listed above which give us the difference tones through their mathematical simplicity.) Now take the **b**" found a different way: going up by 5ths across the open strings and up to the first **b**" above the E string *a la* Pythagoras (3/2 multiplied by itself four times.) This latter **b**" has a fraction to the open G string of 81/16. We need to subtract two octaves from this **b**" to get it down into the range of the **b** above the G string with which we started so that we can compare the two theoretically identical notes and their intervals to the open G string. When adding ratios (intervals) by piling them upwards we multiply their fractions. But when we subtract intervals leading downwards we divide their fractions. And as you remember from your 4th-grade fractions course when you divide fractions you invert the numerator and the denominator of the 2nd fraction and then multiply.

So to get two octaves going up you multiply 2/1 x 2/1= 4/1. But to subtract two octaves going down you invert 4/1 into 1/4 and then multiply. So in this case: 81/16 x 1/4= 81/64. This gives you the ratio of that original **b** to the open G string in its Pythagorean form based on perfect 5^{th}s. But is this the same interval as the Tartinian 5/4?

Now we come to that fun part of acoustical theory where we compare two different intervals and see whether they are identical; or if not, which is the larger and which is the smaller. To make our calculations as easy as possible we try to reduce the two denominators into their lowest possible terms by dividing both terms by the highest possible identical number, in this case, 4. By reducing the two denominators to their lowest possible terms by dividing each by 4 we get: 5/1 vs 81/16. Once having acquired the smallest possible numbers to calculate with we apply the operation known as "cross multiplying." In cross multiplying we multiply the numerator of the first fraction by the denominator of the second and vice versa. This operation looks like: 5 x 16=80 and 81 x 1=81. We then set the two resultant products into a new fraction with the higher result as the new numerator and the lower result as the new denominator. This gives us the acoustical space between the two intervals, here 81/80. Thus we see that the two original intervals, though named by us with the same letter names, are not identical.

So which one is the larger and which one the smaller? Since the "5" with which we multiplied the 16 came from the Tartinian "5/4" its result (80) is the new denominator because it's smaller than that of the other multiplication, 81x1. The "81" came from the 81/64 (originally 81/16) thus the Pythagorean version; and being the larger number becomes the new numerator. Thus the acoustical space between the larger Pythagorean major 3rd and the smaller Tartinian major 3rd is 81/80. Now we could have arrived at the same result by adding two octaves to the Tartinian major 3rd **b** (5/4 x 4/1) rather than subtracting those two octaves from the Pythagorean **b**" (81/16 x 1/4) in this way: 20/4=5/1 vs 81/16, when cross multiplied (81 x 1=81, 5 x 16=80:) thus the same 81/80. 81/80 is known as the "Comma of Didymus" or the "syntonic comma." It plays a highly important and prominent role in many basic acoustical comparisons.

So if the Pythagorean Major 3rd is not 5/4 then what is it? It happens to be that 81/64 mentioned; and not only will that not reduce to any smaller terms nor as a double stop produce any kind of identifiable difference tone to our ears; besides that it will sound highly offensive and dissonant as well. And since our Western notions of music until well into the Middle Ages derived from the Greeks, their assessment that the Major 3rd was a "dissonance" (though true enough in their Pythagorean terms) seems highly unbelievable to us Western Europeans to whom the loveliness of the Major 3rd and its multifarious resulting harmonies seem the basis of all musical beauty. But this was why organum, Western music's first attempts at counterpoint, so primitive-sounding to our modern ears, developed 1,000 years ago on the basis of parallel 4ths, 5ths, and octaves, with the 3rd conspicuously absent. It took many centuries and much well-tempered tuning to smooth away that characteristic Pythagorean dissonance for the 3rd to establish itself as the basis of our normal harmonies.

Now we are ready to go up the fingerboard half step by half step. Of the various named enharmonic alternative possibilities for each half step indicated above only one will be chosen for illustrations of tuning. For example **e**" will be used instead of **f**b" as the equivalent of the open E string, etc. The chosen notes create the most basic and most often used forms of chords and double stops on their respective half steps. Some notes tune directly to an open string; others require (a) pre-tuned finger placement(s) as their tuning reference point(s),"preliminary tunings."

1. 1st half step, high version; leading from E string leftward to G string: **f**", **b**b', **e**b'.

When each of these three notes is tuned to its leftward open string a just minor 6th, 8/5, results giving us the three-part fraction 8/5/3. From this we can see that the resulting difference tones are a major 6th below the lower notes of their double stops; thus: **c, f, B**b respectively. The remaining **a**b has no C string to tune to but is a perfect 5th, 3/2, across from the **e**b'. The resulting difference tone of this tuning is 3/2/1, thus an octave below the lower **a**b, **A**b. Note that all these notes are on the **high** version of the first half step, not the low. The low versions are out of tune to their leftward open strings and are thus not tunable except by ear judgment in scalar/arpeggio, but not double stop, mode. Note also that the minor 6th creates as a difference tone a pitch entirely different by name from the two actual notes being played and completes the major triad started by the two played notes. See also #6 below.

1a. HINT ON TUNING FINGERED 5THS: If the first try at a fingered 5th is out of tune play the two notes involved individually and determine which note is flat. Roll or move your finger toward that note; it will sharpen thus tuning the 5th.

2. 2nd half step, low version: leading leftward from E string to G string: **f#**", **b**', **e**'. When each of these notes is tuned to its leftward open string A, D, and G respectively, a just major 6^{th}, 5/3, results giving us the three-part fraction 5/3/2. Thus we see that the resulting difference tones are perfect 5^{th}s below their lower notes; thus **d**', **g**, **c** respectively. The remaining note **a** has no C string to tune to but is a perfect 5^{th}, 3/2, across from low **e**'. The resulting difference tone of this tuning is 3/2/1, thus an octave below the lower note **a**, **A**. The major 6^{th}, like the minor 6^{th}, produces a difference tone which completes the major triad begun by the two played notes. See #1 above.

3. 2nd half step, high version: leading rightward from G string to E string: **a**, **e**', **b**'. When each of these notes is tuned to its next rightward open string a perfect 4^{th}, 4/3, results giving us the three-part fraction 4/3/1. By doubling the [1] to a "2" for reference

we see that the difference tones are perfect 5ths plus an octave below each of the lower notes; thus **D**, **A**, **e** respectively. The **f**#" has no B string to tune to but is a perfect 5th,3/2, above **b**'. The resulting difference tone of this tuning is 3/2/1, thus an octave below the lower note **b**', **b**.

3a. DIFFERENCE TONE HINT: I tell my students: for a perfect 5th you hear the lower note as a difference tone; for a perfect 4th you hear the upper note.

3b. BASIC EXERCISE FOR THE 2ND HALF STEP: Play **low** version with leftward open string then, without moving the finger, compare that note to the rightward string. The note is much too flat and out of tune and must be moved to the **high** version where it is in tune with the rightward string. Play the high version, again without moving the finger, with the leftward string: much too sharp and out of tune; must be moved back to the **low** version to be in tune. (I usually start the students with the **b**' against the E string followed by the **b**' against the D string; followed up with the **e**' on the D string against its two comparison strings.)

4. 3rd half step, low version, preliminary tuning: leading leftward from E string to G string: **g**", **c**". We start from the right side of the fingerboard by creating a secondary tuning. We form respectively a three-part G Major chord, then a C Major chord. This serves to tune the notes **g**" and **c**". For the G Major chord on the three strings D, A, and E the notes, reading rightward, are the open D string, the low version of the **b'**, and the low version of the **g**". The low **g**" matches (is exactly two octaves above) the open G string: **g**" frequency 782. The difference tone of the open D-**b**', being a major 6th, has been described above in #2. The interval between the **b**'-**g**" is a minor 6th, 8/5, and thus produces a difference tone of 8/5/3. Thus we see that the difference tone is a major 6th below the lower note **b**' which is **d**'. This is the same note as the difference tone 8/3 to the upper note (3x2="6"; 8/"6"=4/3): a perfect 4th plus an octave below the upper note. Thus for the Minor 6th you will hear as a difference tone a third note completely different from the two notes you are playing. This new note will complete the major triad begun by your two played notes: **g**"-**b'** (downwards) creating **d**. Compare to ##1 and 2 above. The notes **f'** and **b**b need to be tuned by perfect 5ths down from the low **c**". They and the **c**" must not be checked against their rightward open strings. This would give us the high version (See next.)

5. 3rd half step, high version: leading rightward from the G string tune the notes **b**b, **f'**, **c**" to their rightward open strings D, A, and E. This will give us in each case the just major 3rd, 5/4. Its difference tone is 5/4/1. Double the [1] to a "2" for reference (5/4/"2"). We see that the difference tone of a major 3rd is two octaves below the lower note (4/2 reduces to 2/1.) Similarly to the octave and the perfect 5th the major 3rd reproduces its lower note as a difference tone, whether by unison or octave(s). The note **g**" on the E string has no B string to tune to and thus needs to be tuned by perfect 5th from the high **c**" with the method described above.

5a. BASIC EXERCISE FOR THE THIRD HALF STEP: Find the low **c**" as above. Without moving the finger compare it to the E string. It is too flat and out of tune. Move the **c**" forward without moving the first finger until the **c**" is in tune with the E string. Then compare that note to the unmoved **e'** on the D string: very sharp and out of tune. Raise the low **e**' until it is in tune with the high **c**" and then compare it to the G string: much too sharp and out of tune.

6. 4th half step, low version: leading leftward across the fingerboard tune the notes **c#**", **f#**', and **b** with their rightward open strings, E, A, and D. This gives us in each case a just minor 3rd, 6/5, whose difference tone is thus 6/5/1 adjusted by two reference octave multiplications to 6/5/"4". This shows us that the difference tone of a minor 3rd is a major 3rd plus two octaves below the lower note or a perfect 5th plus two octaves below the upper note; here the notes **a**, **d**, and **G** respectively. The double stops most rightward on the fingerboard are of course the easiest to hear, being higher in pitch. That's why I start my students from the E string side of the fingerboard. The note, low version **g#**" on the E string, has no B string to tune to so must be tuned as a perfect 5th from the low version **c#**". Alternatively a two-step preliminary tuning can be used: high version **b**' on the A string then the low version **g#**" on the E string as a just major 6th 5/3. See #2 above. The minor 3rd, like the minor 6th in #1, and the major 6th as in #2 above, creates a third pitch as a difference tone entirely different by name from the two actual notes being played and completes the major triad started by the two played notes.

7. 4th half step, high version, preliminary tuning: leading leftward across the fingerboard establish respectively a high version 2nd half step **b**' on the A string, then a similar **e**' on the D string. From these notes downward tune an **f#** on the D string then a **b** on the G string, usually fingered 1st finger-2nd finger. Between the high version 2nd half step and the high version 4th half step is the perfect 4th, 4/3, with difference tone 4/3/1, referenced to 4/3/"2", difference tones similar to the high 2nd half step above, #3, here the tones **b** and **E** respectively. The high versions **c#**" on the A string and **g#**" on the E string need to be tuned by perfect 5ths from the high versions **f#**' and **c#**" respectively.

7a. BASIC EXERCISE FOR THE 4TH HALF STEP: Establish the high version 4th half steps as described above. For the **f#**' lift the finger on the **b**' (usually the 1st finger) and compare the high version **f#**' to the open A string: very sharp and out of tune. Move the **f#** finger (usually the 2nd finger) lower until it is in tune with the A string. Then without moving the 2nd finger re-establish the high **b**' against the E string. Then compare the low **f#** with the high **b**': very flat and out of tune. Perform the analagous operation for the **b** on the G string. Finally, lower the **b**' to match the low **f#**' and then compare it to the E string: very flat and out of tune. Reverse the process and recover both high versions on the two neighboring strings.

8. 5th half step, low version: leading leftward across the fingerboard tune the notes **a**", **d**", and **g**' to their respective leftward open strings A, D, and G. This gives you in each case the perfect octave 2/1 with difference tone 2/1/1. As noted above this difference tone is imperceptible as a separate tone. But the soothing lack of beats assures us that our octave to the open string is in tune! The note **c** on the G string has no C string to tune to and must be tuned as a perfect 5th down from **g**'. Also alternating this low **c** with the open G string gives us the successive perfect 4^{th}s which we flatter ourselves as always being able to hear correctly. Alternatively we can tune this note from the low 2nd half step **e**' on the D string as a major 3^{rd} down. See #4 above.

9. 5th half step, high version, preliminary tuning: establish the high 3rd half step on the A string as described in #4 above, the note **c**", usually a 1st or 2nd finger. From this note tune downwards the perfect 4^{th}, note **g**' on the D string. Compare this note to the open G string: distinctly sharp and out of tune. The high version notes leading rightward **c**', **d**", and **a**" on the strings G, A, and E respectively need to be tuned by perfect 5^{th}s from the original high **g**'. Alternatively the note high version **c**' on the G string can be tuned from a high **f'** 3rd half step on the D string tuned to the A string. Also see #4 above.

9a. BASIC EXERCISE FOR THE 5TH HALF STEP: Establish the high **g**' on the D string as described above keeping the original high **c**" in place. Lower the **g**' to the low version in tune with the open G string. Then compare that note with the high version **c**": very flat and out of tune. Lower the **c**" to be in tune with the low **g**' and then compare to the E string: very flat and out of tune. Reverse the process and restore both high versions on their two neighboring strings.

10. 6th half step, the lowest version: there are at least 4 different versions of the 6th half step, easily measured by the violinist; the lowest: preliminary tuning: leading leftward across the fingerboard establish the low version of the 2nd half step as in #2 above, the notes **f#**", **b**', **e**' on the E, A, and D strings respectively. Double stop to these notes the notes **d#**", **g#**', and **c#**' on the A, D, and G strings respectively (usually 1st and 3rd fingers). This creates the just minor 3^{rd} and its resultant 3-part fraction of 6/5/1, with reference adjustment 6/5/"4". See also #6, above. This tells us that the difference tone of a minor 3^{rd} is a major 3^{rd} plus 2 octaves below the lower note, in this case the notes **b**, **e**, and **A** respectively. Though the second and third of these difference tones are well below the normal violin range the first is easily audible. The note **a#**" on the E string must be tuned as a perfect 5th from **d#**" on the A string. Alternatively establish two preliminary tunings: low 2nd half step **f#**" on the E string; a perfect 4th down to the low version 4th half step **c#**" on the A string; finally a major 6th up to the lowest version 6th half step **a#**" on the E string. This is a tuning which is not often used by violin players.

11. 6th half step, the first higher version, preliminary tuning: substitute the high version of the 2nd half step as in #3 above for the low version as in #2, but this time leading leftward across the fingerboard with the notes **b**' and **e**'. Double stop the notes **g**#' and **c**#' to these respectively. The difference tones **e** and **A** will appear as above in #10. The notes **d**#" and **a**#" can be tuned by perfect 5ths from **g**#' and **d**#" but a more usual way of tuning these is the following: to find **d**#" establish the high 4th half step as in #7 above. From the preliminarily tuned high **f**#' on the D string tune the **d**#" on the A string as a just major 6th with its resulting 3-part fraction of 5/3/2. No reference adjustment is needed. This tells us that the difference tone of a just major 6th is a perfect 5th below the lower note, in this case **b** (as if on the G string). For the **a**#" three preliminary tunings are required: high 2nd half step **b**" on the A string; perfect 5th up to a high **f**#" on the E string; down to a high 4th half step **c**#" on the A string; then from there up the major 6th to the **a**#". This should give us the difference tone of **f**#' (as if on the D string) by the relationships described just above in #11 (usually the 1st finger followed by the 2nd and then the 3rd finger). This sounds much more complicated but is usually quickly accomplished.

12. The 6th half step, 2nd higher version, preliminary tuning: establish the high 1st half step as in #1 above on the notes **b**b' and **e**b' respectively. From these tune the perfect octave on the next rightward string, the notes **b**b" and **e**b", usually 1st and 4th fingers. No audible difference tones, just that soothing lack of dissonant beats. The notes **a**b' and **d**b" on the D and G strings respectively tuned by perfect 5ths down from the now-established **e**b". The old conundrum whether an **a**# is higher or lower than a **b**b is by this method of tuning resolved in favor of the flats: they are the higher versions. But the sharps have one more trick up their sleeve; see next:

13. The 6th half step, the highest version, preliminary tunings: from the high 2nd half step on the A string **b**" as in #3 above tune down by perfect 4ths to the **f**#' and then to the **c**#', usually 1st, 2nd, and 3rd fingers. This resulting **c**#' is higher (by a few "cents" as acoustical jargon has it) than the **d**b' described in #12. (The mathematics are rather complicated; but take my word for it.)

14. The 7th half step, three versions: the low the middle and the high. The low version, preliminary tuning: leading rightward across the fingerboard establish a low 2nd half step as in #2 above on the notes **e**' and **b**' on the D and A strings respectively. Then tune a perfect octave up from these notes on the A and E strings respectively, usually 1st and 4th fingers. This **e**" will be perceptibly (and annoyingly) flat to the E string. This tuning of **b**" on the E string will not be able to produce the usual harmonic on this half step. The notes **a**' and **d**' on the D and G strings respectively tune down by perfect 5ths from this **e**" (but not from their corresponding open strings; that's next). These tunings of the notes e'/e" and b' (an octave up from the low 4th half step **b** as in #6 above) and b" are used quite a lot; these tunings of a' and d' practically never.

15. The 7th half step, middle version: in any order you please tune the notes **e**'', **a**', **d**', to their respective open strings, any finger, hopefully beatless. The note **b**'' on the E string should produce the proper harmonic on that step. This is the easiest finger tuning you get on the violin, followed by the octaves to the leftward open strings on the 5th half step.

16. The 7th half step, high version, preliminary tunings: leading leftward across the fingerboard establish the high versions of **c**'' as in #5. From this note tune downwards the perfect 4ths **g**' on the D string, mathematics and difference tones as described above for perfect 4ths; and **d**' on the G string,. The **d**' on the G string will be decidedly higher than the open D string; and the second of two perfect 4ths with difference tones of **c** and **g** respectively.

16a. BASIC EXERCISE FOR THE 7TH HALF STEP: Find the high **d**' as above. Then move it lower until it matches the open D string. Then build your two fourths rightward and upwards **d**'-**g**' and **g**'-**c**''. You now have a low version 3rd half step **c**'' as in #4 above. Compare to the E string: much too flat and out of tune. Raise the **c**'' back to its high version in tune with the E string and repeat #16 downwards to the high 7th half step **d**' as above.

17. The 8th half step, three versions, low, middle, and high; the low version: play the note **c**''' on the E string, any finger, and tune to the open A string. But you must play it low enough so that the difference tone **d**' appears. Once the **d**' appears play the **c**''' by itself. It is disconcertingly flat and out of tune. Yet when blended with the A string it sounds especially smooth and euphonious. This note represents the natural flatted 7th of the harmonic series. We don't ordinarily use this note in Western classical music, perhaps in jazz. Since the frequency difference between the open A string and this note is much less than that between the open A string and the more normal **c**''' a lower difference tone results: **d**' vs **f**' (See #18 below.) This interval is probably related to one of those missing ratios as mentioned above, probable the 7/6 (2/1 x 7/6 = 14/6) since that one would be closer to being a real minor 3rd 6/5 than 8/7 would be.

17a EXERCISE FOR THE FLATTED 7TH 8TH HALF STEP: Play the harmonics upward on the D string starting from **d**'': **d**''-**a**''-**d**'''-**f#**'''-**a**'''-**c**^{iv}. This is the flatted natural 7th. Match this pitch on the E string when tuning to the A string.

18. The 8th half step, middle position: leading leftward across the fingerboard play the notes **c**''', **f**'', and **bb**' on the E, A, and D, strings respectively successively to their leftward open strings A, D, and G respectively. This creates a minor 10th sharper than #17 but still not as high as #19, below. The mathematics of this one are rather elaborate but theoretically the difference tone lies somewhere between the **d** of #17 and

the **f** of #19. Its ratio is 64/27. It sounds nice and smooth. Not as smooth as #17 but not as rough as #19.

19. The 8th half step, high version: play the same **c**''' against the A string. This time the difference tone is the more normal **f**. This minor 10th is ratioed by adding a perfect octave to a just minor 3rd: 2/1 x 6/5 = 12/5. To find the difference tone we need to extend our fraction to four parts: 12/5/7/2. (The [2] is [7]-[5]; the frequency subtractions work in reverse as well. This often leaves some difference tones between, rather than below, the two played notes.) Thus the primary difference tone of a minor 10th is actually not well perceptible, involving a [7] as it does. But the [2] it creates by subtraction enables us to compare a secondary difference tone to the two original notes. Thus by reference 12/"8"=3/"2" and 5/"4". Thus the difference tone of a minor 10th is a perfect 5th plus two octaves below the upper note or a major 3rd plus an octave below the lower note, here in both cases **f**. Play the **c**''' as high as you can stand it and still call it in tune. This will give you the true **f** as difference tone. Like the minor 3rd, the minor 10th and its difference tone complete a major triad.

20. 9th half step, two versions, low and high; low version: leading leftward across the fingerboard tune the notes **c#**''', **f#**'', and **b**' on the E, A, and D strings respectively to their respective leftward open strings, A, D, and G. This gives you three just major 10ths of 2/1 x 5/4 = 10/4. Our 4-part fraction becomes 10/4/6/2, giving us various combinations: 10/4 = 5/2 =5/"4"; 6/4 =3/2; 4/2 = 2/1; 6/2 = 6/"4" =3/"2". Thus we will get several difference tones. 5/"4" tells us that the primary difference tone is a major 3rd plus an octave below the upper note, thus identical to the lower played notes **a**', **d**', and **g**. 6/4 or 3/2, originally 4/6 or 2/3 tells us that there is a secondary difference tone a perfect 5th **above** the lower note (i.e. in between the two played notes), here **e**'', **a**', and **d**' respectively. These are much harder to hear than the three primary difference tones. These are difference tones of difference tones. 4/2 = 2/1 tells us that there is another secondary difference tone an octave below the lower played note, thus **a**, **d**, and **G**. Finally, 6/2 = 6/"4" = 3/"2" tells us that there is another secondary difference tone a perfect 5th plus an octave below the difference tones **e**'', **a**' and **d**', namely and identical to and thus reinforcing the secondary difference tones **a**, **d**, and **G** above. Whether or not you hear all these difference tones the smooth euphony of the low version is its defining sound. The remaining note on the 9th half step, low version **e**', is tunable by perfect 5th down from low version **b**' on the D string or matching by ear with the low version **e**' 2nd half step on the D string. This note is too low to match the E string and will not make the E string ring by sympathetic vibration.

21. 9th half step, high version, preliminary tuning: leading leftward across the fingerboard tune successively **e**'' and **a**' on the A and D strings with their rightward open strings E and A with any finger other than the 4th. From these notes successively tune the notes **b**' and **e**' on the D and G strings respectively. The **e**' on the G string will ring the open E string by sympathetic vibration. If there were a B string the high **b**' would ring that as well. The high versions of the **f#**'' and **c#**''' respectively can be tuned

up by perfect fifths from high version **b**'. None of the notes **c#**''', **f#**'' or **b**' can be tuned to their leftward open strings because they are too high.

21a. BASIC EXERCISE FOR THE 9TH HALF STEP: Tune the high version **b**' as in #20 above. Then, without moving the finger playing **e**'', lower the **b**' till it matches the G string as in #19. Play this note against the **e**'': very flat and out of tune. Next lower the **e**'' till it matches the low b'. Compare this note to the open E string: very flat and out of tune. Reverse the process by retuning the **e**'' and **b**' to the E string and again comparing the high version **b**' to the open G string: very sharp and out of tune.

22. The 10th half step, two versions, low and high; the low version: leading leftward across the fingerboard tune the notes **d**''', **g**'', and **c**'' on the E, A, and D strings respectively to their leftward open strings A, D, and G successively. The intervals created are perfect 11ths. The ratio of a perfect 11th, which is simply an octave plus a perfect 4th, is 8/3 with difference tone fractions of 8/3/5/2/1... etc. appearing. Again various relationships appear giving us different registers of the notes **d** and **f#**, **g** and **b**, and **c** and **e** respectively. And again: whether or not you hear these particular difference tones clearly or not the smoothness of the euphony tells you that you are in tune. The note low version **f**' on the G string can be tuned by perfect 5th down from low version **c**'' on the D string but not to the D string itself. That's next.

22a. ANOMALY OF INTERVAL ADDITION: Only in the world of acoustics can 8+4=11. Any two or more intervals when added together into larger compounded intervals drop one Arabic number for each interval added compared to normal addition. This is because in interval addition there is no "zero." Whatever letter name you start on counts as "1". Thus when a new interval gets added its starting letter gets merged with the final letter of the previous interval and the space between "zero" and "1" disappears. Example: Add a 2nd, a 3rd, and a 4th. In normal arithmetic this comes out at "9". But in acoustical arithmetic it comes out as "7": **g** up to **a** = a 2nd; **a** up to **c** = a 3rd; and **c** up to **f** = a 4th. But **g** up to **f** = a 7th.

23. 10th half step, high version: leading leftward across the fingerboard tune successively the notes **g**'', **c**'', and **f**' with their rightward open strings E, A, and D respectively. This is a series of just minor 3rds. See low version of the 4th half step, #6 above.

23a. BASIC EXERCISE FOR THE 10TH HALF STEP: Tune up either #21 or #22 above. Then reverse the relevant open strings to tune against. The note which tunes to the rightward string will not match the leftward string and vice versa. Tune to the second comparison string and recompare to the original string. The relevant notes cannot tune to both rightward and leftward open strings simultaneously.

24. 11th half step, two versions, low and high; low version: leading leftward across the fingerboard tune the notes **g#''**, **c#''**, and **f#'** on the A. D, and G strings respectively to their rightward open strings E, A, and D successively. This creates a series of just major 3^{rd}s creating the difference tones **e'**, **a**, and **d** respectively. See #4 above.

25. 11th half step, high version, preliminary tuning: leading leftward across the fingerboard tune the notes **e''** to the E string 1st or 2nd finger, high version **b'** on the D string and then a perfect 4^{th} down to **f#'** on the G string. This **f#** is too high to tune to the D string as in #23. The high versions of **c#''**, **g#''**, and **d#'''** by perfect 5ths up from this high **f#'**.

25a. BASIC EXERCISE FOR THE 11TH HALF STEP: Establish the high **f#** as in #24 above. Lift other fingers and compare this **f#'** to the D string: too high and out of tune. Lower the **f#'** to tune to the D string as in #23. Then tune from this **f#'** the notes **b'** and **e''** leading rightward on the D and A strings respectively. Compare this **e''** to the E string: much too flat and out of tune. Raise **e''** to tune with the E string and repeat #24.

26. 12th half step, two versions, low and high; low version: play in any order with any finger the respective harmonics **e'''**, **a''**, **d''**, and **g'**. Still another easy fingering.

27. 12th half step, high version, preliminary tuning: Leading leftward across the fingerboard establish successively the high versions of the 10th half step as in #22 above, the notes **g''** and **c''** on the A string and D string respectively. From these notes tune down the perfect 4^{th}s **d''** and **g'** on the D and G strings respectively. The **d''** on the D string will be too sharp to the open G and A strings and too high to produce the normal harmonic; the **g'** on the G string will be too sharp for the D string and will not be able to produce the normal harmonic. Tune the high version notes **a''** and **e'''** on the A and E strings respectively by perfect 5^{th}s from the high version **d''** on the D string.

27a. BASIC EXERCISE FOR THE 12TH HALF STEP: Establish the high notes **d''** and **g'** as in #26. Then lower these notes respectively to their harmonic positions and play the harmonics.

So it turns out that we need to keep track of 27 different spots along our fingerboard to fully accommodate the tuning possibilities inherent in playing on a violin (28 if we count the non-tunable low position of the 1st half step). And then by tuning in 5ths we can transfer these 28 spots to any particular string of the four we have. This is a bit more than one half the 53 possibilities of micro-tunings within the octave as discussed in the Wikipedia article above.

A New Actor on our Interval Stage

Now that we've established the various versions of the different half steps and how to tune them up the question arises: which one of the various choices of pitch do we use when we play any particular note? I'm glad you've asked me that question. The answer is: it depends upon which key you're playing in and how that particular key tunes up to which open strings. But before we can go into that problem we need to discuss the basic tuning of the major and minor scales. When we open up that question we will have to introduce an entirely new interval for discussion, the "contracted minor 3rd" or "CMT" (my terminology). What is a CMT and where do we get one?

BASIC EXERCISE FOR THE CONTRACTED MINOR 3RD: On the D string play a high version 2nd half step **e**' and let it ring the E string as well as tuning it to the A string. Then play a low version 5th half step **g**' and tune it to the open G string. The acoustical space between the high **e**' and the low **g'** is a "contracted minor 3rd" whose ratio is 32/27 (non reducible). It differs from our standard minor 3rd 6/5 by (guess what) our old friend the comma of Didymus 81/80 (6/5 x 80/81 = 32/27).

Where do we put this CMT? In my system of tuning it goes between the 2nd and 4th steps as well as the 6th and 8th steps of any major scale; or to put it into solfeggistic terms between the *re* and the *fa* as well as between the *la* and the *do*. And since all relative minor scales begin on the *la* of the relative major scale our CMT should show up between the 1st and 3rd steps of any minor scale as well. This is vital to the intonation not only of our scales in their linear forms but also in their double stop forms such as scales in 3rds. Not all 3rds, whether major or minor, are created equal, nor do they all produce euphonious difference tones. Our scales in 3rds are mixtures of "just" major and minor 3rds, CMTs, and EMTs (See below.)

For example let's take D Major. As we violinists are wont to do we tune up the Ds of D Major to our open D strings. (What else?) Then we automatically assume that the Es all tune to the open E string. Then we all assume that our Gs tune to the G string. There's our first CMT as described above between the *re* and the *fa* of D Major. Now how about the B? Some of us tune that B to the D string. Sounds reasonable: all the parts of our scale ought to be in tune with its main note, or tonic. Others of us however find more comfort in a **b**' that's tuned to the E string. That also sounds reasonable: if we have a triad on E (as we often do) then its root and 5th ought to be in tune with one another (*re*-*la*). (The **b**'-E string tuning, *la*-*re*, or perfect 4th, is the inversion of the perfect 5th, *re*-*la*.) Now as we noted in ##2 and 3 of our fingerboard studies above, the **b**' which tunes to the D string and the **b**' which tunes to the E string are two different notes, the low and high versions respectively of the 2nd half step. So which note do we choose for our D Major scale? If we choose the high version then we have another CMT between the *la* and the *do* of the D major scale, or any other major scale built along this line.

Now it's time to return to our age-old battle between the Pythagoreans and the Tartinians to decide that question: which version of the *la* do we choose? Or, along which line **do** we build that major scale?

Let's take the Tartinian version first. The Tartinians believe that a major scale is based on three just major triads built on, respectively, the *do*, the *fa*, and the *sol*, or as music theoreticians put it, the I, IV, and V chords. What is a just major triad? It is a triad built on the just intervals, the major and minor thirds 5/4 and 6/5 respectively, conjoined into one entity. A just minor triad reverses the two components, 6/5 conjoined to 5/4. Every triad consists of a root, a 3rd, and a 5th. So we're measuring acoustically the spaces between the root and the 3rd, then the 3rd to the 5th. But every triad, whether major or minor must have a perfect 5th between the root and the 5th, 3/2.

Now the Pythagorean version. The Pythagoreans believe we already have four of the notes of our D Major scale built right into the instrument itself, namely the perfect 5ths making up the four open strings, G, D, A, and E. They would simply extend the series of 5ths to include B, F#, and C#. As long as we're still in D Major we'd have to stop there: the next 5th would be G# and we'd have modulated to A Major. But this D Major example would serve as their model for all major scales.

Now if we take those last three 5ths, **b**'', **f**'''#, and **c**#iv out of their stratospheric E string positions and put their octave equivalents back down into the first position version of the scale on the D and A strings we'd get a series of triads highly "un-just" in nature. Instead of three just triads on I, IV, and V each of which is tuned 5/4 + 6/5 we'd get instead a series of chords looking like "81/64 + 32/27" Remember the Pythagorean major 3rd we discussed above? That's our "81/64". I call it an "expanded major 3rd" or "EMT". Adding an EMT to a CMT still gives us our necessary perfect 5th, 3/2, from root to 5th to form our triad: 81/64 x 32/27 reduces to 27/18 which = 3/2. Both the Pythagorean major 3rd 81/64 (EMT) and the Pythagorean minor 3rd 32/27 (CMT) each differ from their just counterparts 5/4 and 6/5 respectively by the comma of Didymus or 81/80.

So let's apply the two theories to our D Major scale (which serves as a model for all major scales) and see what we get. Both versions are compatible as long as we stick to those four open string notes G, D, A, and E. They are the IV, I, V, and II of our scale respectively. But as soon as we start building triads on these steps the contradictions become apparent. For example, most of us would be inclined (at least at first) to tune our **f**#' on the D string to the A string by choosing the low position of the 4th half step as in the original example in #4 above. This choice would give us the just tonic D Major chord. I still remember my above-mentioned teacher Mischakoff always urging me to "keep that **f#** low" to be in tune.

A similar relationship would exist in the major V triad built on the open A string with a low **c#**'. But it's the IV chord where things get tricky. That's where our choice of Bs comes in as discussed above. We'd normally tune our IV chord on **g**', to the G string. But when we go about building a just major triad on this note we wind up with the **b**' on the low position of the 2nd half step out of tune to the E string as in ##2 and 3 above.

IMPORTANT NOTE ON TUNING: IT'S ALWAYS THE *LA* OF THE SCALE WHICH DIVIDES THE TARTINIAN FROM THE PYTHAGOREAN TUNING.

Most of us violinists prove ourselves true Pythagoreans when we tune our high position 2nd half step **e**' to our **b**' (usually 1st finger, 1st position) as a perfect 5th. Both notes are then in tune to the E string. This also gives us the second of our CMTs in the scale, between the *la* and the *do* **b**'-**d**'. Therefore for many years I have been advising my students:

RAISE THE 6TH DEGREE OF THE SCALE (*LA*).

This followed many years of my trying to be a good Tartinian by keeping the IV chord just along with the I and the V chords. But after a long time trying I gave up and became a Pythagorean on this step of the scale. And you wouldn't believe what a difference this makes in correcting my students' out-of-tuneness. Most of them must be subconscious Tartinians because once they've corrected their 6th degrees they sound very much better. And it takes constant reminding as they perform in different keys to locate and identify that 6th degree so they can know which note to raise. Once they've found it and raised it suddenly they're in tune. Try it on your students.

Pythagoreanism on Steroids

Now that we've settled that controversial **b**' in favor of the Pythagoreans (the 6th degree, still in D Major) let's consider the **f#**' and the **c#**' as well. Going back to the 1st position if we establish our high 4th half step **f#**' (See #6 above.) as 81/64 to the D string and compare it to the A string we get our CMT 32/27 between the III and the V, *mi* and *sol*, of our scale. Yuck! Nobody wants that! But then an amazing thing happens. As soon as we speed up the tempo of our scale to any degree, that just major third **f#**'-**a**' we were so enamored of in its static, unmoving mode (Mischakoff's "low" **f#**', ditto that analagous VII, ti, c#" on the A string) BEGINS TO SOUND TOO FLAT!

This brings us to one of the most remarkable discoveries in the field of acoustics ever made as propounded by that very original genius Theodore Podnos in his book cited above, *Intonation for Strings, Winds, and Singers*: good intonation on the violin varies and is relative to the speed of the music. Podnos gave us the acoustical equivalent of Einstein's theory of relativity which states (among other things) that "the faster you go the heavier you get; and that if you could go as fast as the speed of light you would weigh as much as the entire universe." (My pops formulation.) Podnos' analogy: "the faster you go the more Pythagorean your intonation must become." Meaning? While keeping perfect 4^{th}s and 5^{th}s secure keep raising those major 3^{rd}s until they reach Pythagorean, not "just," dimensions. Thus: raise the *mi* and the *ti* of your major scale since these are the 3^{rd}s of your I and V triads respectively. (The IV triad we've already discussed.) But especially when the tempo picks up. "Just" triads can be well-tolerated and even preferred as long as the chords involved are sustained and the music is not traveling anywhere. But when that train leaves the station put on those Pythagorean traveling shoes! When it comes to my own students I can assure you that this "rule of toe" has solved as many intonation problems as the raised *la*.

But this is not news to most of us. The great string players starting with Casals advised this. They called it "expressive intonation": that *mi* wants to go to that *fa* and that ti wants to go to that *do*: squeeze those half steps! Even in the 1930s the noted acoustician Carl E. Seashore in his famous book, *The Psychology of Music* (1938) interviewed a number of famous performers such as Fritz Kreisler and to a person they all preferred Pythogorean intonation (as do I as revised by Podnos). But today I have a colleague in the authentic performance practice field who insists that Pythogorean intonation is a Romantic latecomer and that violinists back in Tartini's time all used just intonation. And he performs his 18th-century works using those flat-sounding *mi*s and *ti*s as part of the authentic sound of 18th-century music.

On the Intonation of Minor Scales

Minor scales offer certain specialized problems of their own which can be enumerated here. Though the relative minor scale is somewhat based on the intonation relationships of its relative major nevertheless, differences arise.

1. To begin with, I advocate preserving the Pythagorean relationship of the *la-do* as a 32/27. (Relative minor scales begin on the *la* of their relative majors.) This does not seem to be a widespread taste. On recordings of famous violinists of today I hear

usually the 6/5 normal minor 3rd being performed as part of minor scales. This seems awfully sharp to me. My version is more "expressive" in the Romantic tradition.

2. Secondly, in order to maintain a perfect 5th from tonic to dominant the *mi* of the scale must be raised to its Pythagorean position if it had previously been tuned in the Tartini manner: a "just" *mi* will be flat to the tonic of the minor scale; not a perfect 5th. So already a difference between the natural, relative minor and its relative major has arisen if the performer has chosen to perform the relative major in a Tartinian style. (By the way, a reminder that the "natural minor" is simply the relative major starting on the 6th degree without any changes or accidentals.)

3. Third, in my hearing of the minor scale the *re* of the scale (the 4th degree), contrary to the relative major, does not match the tonic as a perfect 4th and if it does it sounds too flat. If raised a bit this seems to improve the intonation. I believe this is based on the following calculation:

9/8 (*la-ti*) x 6/5 (*ti-re*) = 27/20

27/20 is 81/80 (remember that one?) larger than a true perfect 4th, 4/3. 9/8 is the large whole step (major 2nd). But a true perfect 4th, 4/3, seems to be made up of a small whole step, 10/9, not the large, plus the just minor 3rd, 6/5. The raised *re* plus the low-sounding *do* gives for me just the right color to a minor scale.

4. When descending the melodic minor scale in the natural form watch that the 7th degree of the scale, *sol*, is only a small whole step, 10/9, down from the tonic, *la*, rather than a large one, 9/8, otherwise it sounds too flat. I can't prove this acoustically, it's just a preference. But I and the students do sound better if this is done. Mischakoff again: "When you come down your scales, put the fingers forward."

Hidden Accidentals both Decorative and Modulatory, and Hidden Modulations

An interesting and little remembered fact of musical history is that back in J.S. Bach's time the rule of accidentals and their notation was exactly the opposite of what it is today. Nowadays if an accidental is placed in front of a note at the beginning of a bar it is understood to affect every instance of that note within that same bar. Only the next bar line serves to cancel the accidental. (Whether the accidental is valid in other registers of that same note is still sometimes a matter of dispute.) However when Bach wrote his music the rule was that **unless** you placed the relevant accidental each and every time in front of that same note it didn't count, even within the same bar. You can find the perfect examples of these instances in the facsimiles of the unaccompanied sonatas and partitas as republished in the Galamian Edition. I suspect that it was later copyists and typesetters who got tired of placing all those extra accidentals in place who established the modern rule, perhaps around 1800. This changed rule of accidentals has been bedeviling sight readers ever since. By the time the sight reader reaches the end of the bar and the accidentals have disappeared it is ever so easy to forget that they were ever there in the first place. These are the HIDDEN ACCIDENTALS! Therefore my first rule to my students when they first begin to study their parts is:

PENCIL IN ALL THE HIDDEN ACCIDENTALS. IF IT WAS GOOD ENOUGH FOR BACH IT'S GOOD ENOUGH FOR YOU.

Accidentals come in two broad categories, decorative and modulatory. The first thing a good sight reader must establish for him/herself as she/he reads is which is which. A decorative accidental serves to create a note which embellishes a normal step of the scale of the key established by the key signature without changing the key. Modulatory accidentals add or subtract sharps, flats, and naturals to notes which take the music to keys other than that of the key signature. For example, a piece printed with a key signature of two sharps is going along without accidentals when suddenly a bunch of **g**#s begin to appear. Are these **g**#s decorating the 5th degree of D major or are they taking the music to A Major? This decision is important for intonation. In general it's a good idea to squeeze the decorative half steps as close as possible to their main notes ("expressive intonation"). If a new key has appeared you treat it as you would with the Tartinian or Pythagorean methods you favor. In the example of the **g#** above, if it were decorating the *sol* of D you'd want to squeeze it closely to **a**. But if that **g#** now represents the *ti* of your A Major scale it's up to you to choose whether to squeeze it close to **a** (Pythagorean) or tune it to **e** as a just major 3^{rd} (Tartinian).

If a new key has indeed been put into place, composers, following the modernized rule of accidentals, often like to trick us with HIDDEN MODULATIONS. Suppose you're now going along in that new key of A Major complete with its necessary **g#** accidentals when the composer decides he'd like to go back to D Major after all. By now that g# has become second nature to you and, key signature or no key signature, you keep supplying it to the music. Suddenly a bar line appears and the previously accidentalized **g** returns to its original unsharped condition. However since there is no **g#** in the key signature the composer feels he doesn't have to treat this event as the modulation it really is. And there we are: before we know it we're playing that **g** unconsciously and wrongly as a **g#** simply out of inertia and habit. Therefore my next rule of sight reading is:

PENCIL IN ALL THOSE NATURALS WHICH TAKE YOU BACK TO THE KEY SIGNATURE EVEN THOUGH BY NOTATION RULES THEY'RE NOT REQUIRED TO BE THERE.

When to use Which Notes

All right, having isolated all those various notes and shadings how do we put them to use to create good intonation? Here are my recommendations:

1. With the exception of the Pythagorean raised *la* of the scale I orient myself in the Tartinian mode to begin with. This creates many pure just triads and double stops and lets the violin ring lustrously. But I'm ready to become Pythagorean when the tempo picks up.

2. Determine which key you are performing in. This includes temporary modulations within a piece.

3. When you add sharps to the key signatures you are raising the pitches necessary to play in tune and you will be choosing your notes more and more often (but not always) from the high versions available. Adding flats sends you more and more often (but again, not always) to the lower versions. This is because:

4. If you are thinking Tartinially with a low *mi* for your scale then every time you add a sharp, this *mi* is transformed into your new *la* and thus must be raised. When you add flats a previously high *la* must serve as the new *mi* and thus needs to be lowered. So by the time you've added all your sharps every note has been raised and by the time you've added all your flats all your notes have been lowered. Example: the **b** of G Major, one sharp, low *mi* in Tartini terms, must be raised to become the new *la* in D Major, two sharps, Pythagorean style. And so on with the notes **f**#, **c**#, **g**#, **d**#, **a**#. In F Major, one flat, the note **d**, which is the *la*, is the original high position 7th half step of #16 higher than the D string which needs to be lowered to the middle 7th half step, the *mi* of Bb Major, #15 which is in tune to the D string. And so on with the notes **g**, **c**, **f**, **b**b, **e**b.

Speaking of keys, most of us violinists realize that the best (easiest, most brilliant) key on the violin is D Major. That's why it has been so favored among composers down through the years. (Followed in favor by G, A, and E, our other open strings.) However not all of us realize that the hardest key on the violin is the easiest key on the piano, namely C Major (on the viola and cello F Major). And that another easy key on the piano, F# Major, is among the hardest on the violin. Why is this?

The fact is that the basic triad of C Major, C, E, and G, includes two notes, G and E, which correspond to two of our open strings, IV and I. However, these two notes, Pythagoreanly tuned in perfect 5ths G-D-A-E do not match each other as pure just intervals Tartinially speaking. This you will have discovered through the ##1-27 variations listed above along with their different exercises. So what do we violinists have to do?

Down through the centuries various remedies have been applied to try to resolve this dilemma, including for violinists to raise the G string and lower the E string till they match as just intervals; and for violists and cellists to keep the C and G strings as a perfect 5th but to raise these both compared to the D and A strings, also kept as a perfect 5th, hoping that the bruised 5th between G and D might not come to anyone's attention too often. In this way they manage to create a C string which is in tune with the violin's E string (good for string quartets).

Now the big problem of C Major (violists and cellists: F Major) is that you can tune the root of your basic tonic triad, **c**, in two different ways, one low, one high, and you have to choose between the two depending on the musical context and often shifting between one and the other. Are we tuning to the G string or the E string (violists, cellists: C string or A string)? It depends upon which register your music is written in. Second violinists, whose music is more usually written in their lowest registers, are constantly tuning to the G string while first violinists, whose music is predominantly in the higher registers, are usually tuning to the E string. This gives most first violinists the impression that second violinists are always playing flat and it gives second violinists (and violists and cellists) the impression that first violinists are always playing too sharp (which they are, relative to one another). So most of the time seconds, violists, and cellists are having to raise their performed pitches to try to match the first violins. It's not too often that the process is worked in reverse: it's less brilliant that way.

If you tune your **c** to the G string and start up your scale from there, whichever mode you're in, whether Tartinian or Pythagorean you're pretty much all right until you reach c". Then if you're still thinking Tartini you must raise your c" to match the E string. If you're in Pythagorean mode however, that high-sounding open E string might be just what you want. If you're playing your **c** chords in the upper positions you'll want to tune them to the E string. This will leave your G string sounding too flat.

But the big problem concerns the other steps of the scale especially the *re*, the *la*, and the *sol*, the notes **d**, **a**, and **g**. If you tune your **c**" in its Tartinian high position on the A string then YOUR DS, AS AND GS WILL ALL BE HIGHER THAN THEIR CORRESPONDING OPEN STRINGS on their respective high versions of their various half steps such as d' (high 7th, high 5th, high 10th), g' (high 5th, high 10th, high 12th), and a' (high 7th, high 12th), etc. Similar problems continue in lessening degrees through the first two flat keys, F Major and Bb Major. Not everyone realizes this. Almost everyone is trained from the outset to tune everything to their open strings. In this case a disastrous procedure. Still with Mischakoff: "When you slide to the 3rd position play everything higher." He was obviously thinking of those **c**s, **g**s, **d**s, and **a**s, our usual first fingers in the 3rd position, as sounding too flat when compared to their respective open strings. Again, it took me many years to come up with the acoustical laws which stood behind these words of pragmatic intonational wisdom

So let's try a few examples naming the half step versions appropriate in each case. Re-consult ##1-27 to remind yourself which are which. Choices: T=Tartinian, P=Pythagorean. These are lists only of your basic scale steps. When it comes time to add decorative passing tones and embellishments to these basic notes squeeze those half steps as closely as possible as described above.

G Major from the G string, 1st position: **a** high; **b** low T, high P; **c**' low; **d**' middle

D string: **e**' high; **f#** low T, high P; **g**' low; **a**' middle

A string: **b**' low T, high P; **c**'' low; **d**'' low; **e**'' middle

E string: **f#**'' low T, high P; **g**'' low; **a**'' low; **b**'' low T, high P

G string, 3rd, 4th positions: **e**' high; **f#**' low T, high P; **g**'' low

D string " **b**' low T, high P; **c**'' low; **d**'' low

A string " **f#**'' low T, high P; **g''** low; **a''** low

E string " c''' middle; d''' low; e''' low, f#''' low T, high P; g''' low

F Major from the G string, 1st position: **a** high; **b**b low; **c'** high (This assumes that your tonic f' is tuned to the A string as it should be.); **d**' high (If you play your scale with the open D string use the low c. Note that if you base your scale on an f' tuned to the A string on its high position of the 3rd half step, your open G string is too flat.)

D string: **e**' high; **f**' high; **g**' high; **a**' middle

A string: **b**b, high; **c**'' high; **d**'' high (higher than the D string); **e**'' middle

E string: **f**'' high; **g**'' high; **a**'' low; **b**b 2nd higher (#12)

If you want to get a more Pythagorean sound, base your tonic **f'**s and your dominant **cs** on their low versions. Then your open A and E strings will give you the sharp-sounding alternate versions of your *mi*s and your *ti*s.

One could finish this series of intonation charts all the way to the end if one had the time and maybe someday I will. In the meantime good luck to you all in playing your string instruments in tune.